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Conditional Expectations in von Neumann Algebras and a Theorem of Takesaki

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1. INTRODUCTION

Conditional expectations play an important role in classical probability theory. In the general context of von Neumann algebras they were implicitely used by von Neumann [41, Chap. II] and by Dixmier [14]. Nakamura and Turumaru [27] and Umegaki [36–39] introduced an axiomatic definition of the concept of conditional expectation in the framework of von Neumann (or C^*-) algebras and established many properties of these objects especially in the context of von Neumann algebras with a finite trace. Their starting point was the characterization, given by Moy [26], of the classical conditional expectations as operators on spaces of measurable functions. Tomiyama showed [33] that conditional expectations, in the sense of the above mentioned authors, can be characterized as norm one projection in C^*-algebras. The importance of norm one projection in the classification problem of von Neumann algebras was recognized by Hakeda and Tomiyama [23] and subsequent research on this argument confirmed the usefulness of these objects. This line of thought culminated in the fundamental work of Connes [12] in which approximately finite von

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Neumann algebras acting on a (separable) Hilbert space $\mathscr{H}$ are characterized as those von Neumann algebras which are the range of a norm one projection from $\mathscr{B}(\mathscr{H})$ (= all bounded linear operators on $\mathscr{H}$). Thus the norm one projections are nowadays a very important tool in the theory of C^*- and W^*-algebras.

However, in spite of the striking algebraic similarity, there is a difference between classical conditional expectations and norm one projections which challanges the candidature of the latter ones as the appropriate generalization of the former. In fact, if $\mathfrak{A}$ is a von Neumann algebra, φ a normal faithful state on $\mathfrak{A}$ and $\mathscr{B}$ a von Neumann sub-algebra of $\mathfrak{A}$, in the classical case (i.e., when $\mathfrak{A}$ is abelian) there is a unique normal faithful conditional expectation E from $\mathfrak{A}$ onto $\mathscr{B}$ satisfying $\varphi = \varphi \cdot E$, while if $\mathfrak{A}$ is a generic von Neumann algebra a normal faithful norm one projection with these properties rarely exists. More precisely, a theorem of Takesaki [31] (independently proved also by Golodez [20]) asserts that such a norm one projection exists if and only if the sub-algebra $\mathscr{B}$ is left globally invariant by the action of the modular automorphisms group of $\mathfrak{A}$ associated to φ.

The inadequacy of norm one projections as analogues of classical conditional expectations was also pointed out in connection with some problems arising in physics, for example, in the theory of the quantum measurement process [13] and in the quantum theory of coarse-graining [21], and in mathematics, namely, in the theory of quantum stochastic (more specifically, quantum Markov) processes; it motivated various attempts to introduce generalizations of the concept of contitional expectations which went beyond the frmework of norm one projections. In particular, some heuristic computations on matrix algebras suggested a natural candidate for this generalization [1].

The main result of the present work consists in giving an alternative characterization (with respect to the Doob–Moy characterization mentioned above) of the classical conditional expectation E associated to a given normal faithful state φ from a (commutative) von Neumann algebra $\mathfrak{A}$ onto a sub-algebra $\mathscr{B}$, in proving that the properties which characterize E in the classical case uniquely define, in the case of arbitrary von Neumann algebras $\mathfrak{A} \supseteq \mathscr{B}$ and faithful normal state φ on $\mathfrak{A}$, a completely positive identity preserving faithful normal map $E: \mathfrak{A} \to \mathscr{B}$ such that $\varphi \cdot E = \varphi$, in the derivation of the explicit form of this map (cf. Theorem (3.5)).

The idea at the basis of our characterization of the classical φ-conditional expectation E is very simple: E is a completely positive identity preserving faithful normal map from $\mathfrak{A}$ to $\mathscr{B}$ such that $\varphi \cdot E = \varphi$, and any such a map defines a unique map E' from the commutant of $\mathscr{B}$ to the commutant of $\mathfrak{A}$ with the analogous properties (cf. Sect. 2, for the notations, and Proposition 3.1). We call E' the φ-dual of E. In the classical case E induces a partial isometry, with the cyclic space of $\mathscr{B}$ as final space, on the GNS space of $\{\mathcal{A}, \varphi\}$; we show, in the general case, that E has this property if and only if E' is an embedding (cf. Proposition 3.3). But the Tomita conjugate isomorphisms $j_{\mathcal{A}}: \mathcal{A} \to \mathcal{A}'$, $j_{\mathcal{B}}: \mathcal{B} \to \mathcal{B}'$ set up a one to one correspondence between embeddings $E': \mathcal{B}' \hookrightarrow \mathcal{A}'$ and embeddings $E'' = j_{\mathcal{A}}^{-1} E' j_{\mathcal{B}}: \mathcal{B} \hookrightarrow \mathcal{A}$. If E' is the φ-dual of E we will call E'' the φ-bidual of E (and E the φ-bidual of E''). The classical φ-conditional expectations is characterized, among all the φ-biduals of embeddings $\mathcal{B} \hookrightarrow \mathcal{A}$ by the property:

$$E \text{ is the } \varphi\text{-bidual of the identity embedding } \iota: \mathcal{B} \hookrightarrow \mathcal{A}. \tag{CE}$$

It is therefore natural to take condition (CE) as the definition of the φ-conditional expectation also in the case in which $\mathcal{A} \supseteq \mathcal{B}$ are arbitrary von Neumann algebras and φ a faithful normal state on $\mathcal{A}$ (for the generalization to weights cf. Sect. 7). In Section 5 we show that the fixed point algebra of E is the largest sub-algebra of $\mathcal{B}$ which satisfies the above mentioned condition of Takesaki. Thus if $\mathcal{B}$ itself satisfies this condition the φ-conditional expectation defined by (CE) and the one defined by Takesaki's theorem coincide. In the case of matrix algebras, the expression we find for E coincides with the one suggested in [1] and, in the semifinite case, it gives a meaning to the natural heuristic extrapolation of this expression (cf. equality (3.28)). The considerations above confirm the intuition, coming from classical probability theory, according to which the φ-condition expectation is a tool for controlling the "statistical location" of a von Neumann algebra $\mathcal{B}$ inside a larger von Neumann algebra $\mathcal{A}$ with respect to a given faithful normal state φ on $\mathcal{A}$. This is to be distinguished from the algebraic location of $\mathcal{B}$ in $\mathcal{A}$ which is independent of any state φ on $\mathcal{A}$.

2. Notations

The notations established below will be used, unless explicitely stated, throughout the paper (with the exception of Sect. 7). $\mathcal{A}$ and $\mathcal{B}$ will denote two von Neumann algebras such that $\mathcal{B} \subseteq \mathcal{A}$. We assume that both $\mathcal{A}$ and $\mathcal{B}$ act on a complex Hilbert space $\mathcal{H}$ with scalar product $\langle \cdot, \cdot \rangle$ and that $\Phi \in \mathcal{H}$ is a cyclic and separating (unit) vector for $\mathcal{A}$. The faithful state on $\mathcal{A}$ defined by Φ will be denoted φ (i.e., $\varphi(a) = \langle \Phi, a\Phi \rangle$) and its restriction to $\mathcal{B} - \varphi_0$. For any sub-set $\mathscr{C} \subseteq \mathcal{A}$, $[\mathscr{C} \cdot \Phi]$ denotes the closed Hilbert space generated by the set $\mathscr{C} \cdot \Phi = \{c\Phi: c \in \mathscr{C}\}$, and the orthogonal projection onto $[\mathscr{C} \cdot \Phi]$ will be denoted $P_{[\mathscr{C} \cdot \Phi]}$. The space $[\mathcal{B} \cdot \Phi]$ will be sometimes denoted $\mathcal{K}$ and the orthogonal projection onto $[\mathcal{B} \cdot \Phi]$ will be simply denoted P (if no confusion is possible). $\mathcal{A}'$ denotes the commutant of $\mathcal{A}$ in $\mathcal{H}$ and $\mathcal{B}'$ the commutant of $\mathcal{B}$ in $\mathcal{K} = [\mathcal{B} \cdot \Phi]$. The state induced by Φ on $\mathcal{A}'$ (resp. $\mathcal{B}'$) will be denoted φ' (resp. φ_0'). When no confusion is possible we will not distinguish between the actions of $\mathscr{B}$ on $\mathscr{H}$ and on $[\mathscr{B} \cdot \Phi]$ (i.e., we will use the same symbol, say, b, for an element of $\mathscr{B}$ considered as an operator on $\mathscr{H}$ and for the same element considered as an operator in $[\mathscr{B} \cdot \Phi]$). If H is an operator acting on $\mathscr{H}$ its domain is denoted $\mathscr{D}(H)$ and its adjoint H^+, while the adjoint of elements $a \in \mathscr{A}$ or $a' \in \mathscr{A}'$,..., are denoted a^*, a'^*,... . The closure of the operator $a\Phi \mapsto a^*\Phi$ ($a \in \mathscr{A}$) will be denoted $S_{\mathscr{A}}$ and its adjoint $F_{\mathscr{A}}$. $F_{\mathscr{A}}$ is the closure of $a'\Phi \mapsto a'^*\Phi$ ($a' \in \mathscr{A}'$), and one has the polar decompositions

$$S_{\mathscr{A}} = J_{\mathscr{A}} \Delta_{\mathscr{A}}^{1/2} = \Delta_{\mathscr{A}}^{-1/2} J_{\mathscr{A}}; \qquad F_{\mathscr{A}} = J_{\mathscr{A}} \Delta_{\mathscr{A}}^{-1/2} = \Delta_{\mathscr{A}}^{1/2} J_{\mathscr{A}}.$$

$\Delta_{\mathscr{A}}$ is a positive invertible operator, called the modular operator associated to φ, and $J_{\mathscr{A}}$ is a anti-unitary involution—the Tomita involution associated to φ. The fundamental results of the Tomita–Takesaki theory are that

$$J_{\mathscr{A}} \mathscr{A} J_{\mathscr{A}} = \mathscr{A}'; \qquad \Delta_{\mathscr{A}}^{it} \mathscr{A} \Delta_{\mathscr{A}}^{-it} = \mathscr{A}; \qquad t \in \mathbb{R}.$$

The one-parameter automorphisms group of $\mathscr{A}$ defined by $\Delta_{\mathscr{A}}^{it}(\cdot)\Delta_{\mathscr{A}}^{-it}$ is called the modular automorphisms group and denoted $\sigma_{\mathscr{A}}^t$ (for the properties of $\Delta_{\mathscr{A}}$, $J_{\mathscr{A}}$, $\sigma_{\mathscr{A}}^t$ we refer to [30], or to the more succint exposition [32, 40]). The analogous objects $S_{\mathscr{B}}$, $\Delta_{\mathscr{B}}$, $J_{\mathscr{B}}$, $\sigma_{\mathscr{B}}^t$,... for $\mathscr{B}$ are defined with respect to the Hilbert space $[\mathscr{B} \cdot \Phi]$.

A linear map $\alpha: \mathscr{A} \to \mathscr{B}$ will be called φ-compatible if $\varphi = \varphi \cdot \alpha$. By a completely positive identity preseving map between two von Neumann algebras we mean a linear completely positive map which transforms the identity of one algebra into the identity of the other. The symbol 1 will denote the identity of the algebra we are speaking about; when some confusions might arise we write $1_{\mathscr{A}}$, $1_{\mathscr{B}}$,... to mean the identity in $\mathscr{A}$, $\mathscr{B}$,... .

The term embedding will be used to mean a normal, injective *-homomorphism.

3. Conditional Expectations as Biduals of Embeddings

In the following we shall frequently use some essentially known facts (cf. [4]) which we sume up in:

Proposition 3.1. *Let $\mathscr{M}$ and $\mathscr{N}$ be W^*-algebras acting repectively on the complex Hilbert spaces $\mathscr{H}$, $\mathscr{K}$ with cyclic and separating vectors Φ and Ψ. Let $F: \mathscr{M} \to \mathscr{N}$ be a positive map such that*

$$\psi \cdot F = \varphi \tag{3.1}$$

(where $\psi(n) = \langle \Psi, n\Psi \rangle$, $\varphi(m) = \langle \Phi, m\Phi \rangle$, $n \in \mathcal{N}$, $m \in \mathcal{M}$*). Then there exists a positive linear map* $F' : \mathcal{N}' \to \mathcal{M}'$ *uniquely defined by*

$$\langle n'\Psi, F(m)\Psi \rangle = \langle F'(n')\Phi, m\Phi \rangle, \qquad m \in \mathcal{M},\ n' \in \mathcal{N}'. \tag{3.2}$$

F' *is normal and, if* F *is completely positive such is* F'*. Moreover if* $F(1) = 1$ *then* $F'(1) = 1$ *and* F' *is faithful.*

Remark. Using (3.2) one easily shows that any F satisfying (3.1) is normal.

Proof. For each $n' \in \mathcal{N}'_+$ the linear map $m \in \mathcal{M} \mapsto \chi(m) = \langle n'\Psi, F(m)\Psi \rangle$ satisfies $\chi(m) \leqslant \|n'\| \, \varphi(m)$, for each $m \in \mathcal{M}_+$. Hence [14, p. 48] there exists an element $F'(n') \in \mathcal{M}'_+$ which satisfies (3.2) for every $m \in \mathcal{M}$. The map $n' \mapsto F'(n')$ can be extended by linearity on the whole of $\mathcal{N}'$, an relation (3.2) is clearly preserved.

By construction F' is positive. Let (n'_α) be a net in $\mathcal{N}'_+$ with $\sup n'_\alpha = n'$, then if $\sup F'(n'_\alpha) = m'_0$ one has for each $m \in \mathcal{M}$,

$$\sup_\alpha \langle F'(n'_\alpha)\Phi, m\Phi \rangle = \langle m'_0 \Phi, m\Phi \rangle$$

and, using (3.2),

$$\sup_\alpha \langle F'(n'_\alpha)\Phi, m\Phi \rangle = \langle F'(n')\Phi, m\Phi \rangle.$$

Since Φ is cyclic and separating this implies that $F'(n') = m'_0$ and F' is normal.

If $k \in \mathbb{N}$; $n'_1, ..., n'_k \in \mathcal{N}'$; $m_1, ..., m_k \in \mathcal{M}$, then:

$$\begin{aligned}
\sum_{i,j=1}^{k} &\langle m_i \Phi, F'(n_i'^* n_j')\, m_j \Phi \rangle \\
&= \sum_{ij} \langle m_j^* m_i \Psi, F'(n_i'^* n_j')\Psi \rangle \\
&= \sum_{ij} \langle F(m_j^* m_i)\, n_i' \Psi, n_j' \Psi \rangle
\end{aligned}$$

and since both Φ and Ψ are cyclic and separating this is sufficient to establish the equivalence between the complete positivity of F and of F'.

$F(1) = 1$ implies $F'(1) = 1$ because of (3.1) and the fact that Φ is cyclic and separating. Finally from (3.2) and the fact that Ψ is separating follows that if $F(1) = 1$ then F' is faithful.

Remark. Denoting ψ', φ' respectively the states induced by the vectors Ψ, Φ on $\mathscr{M}'$, $\mathscr{N}'$, one has

$$\varphi' \cdot F' = \psi' \tag{3.3}$$

which establishes a complete duality between F and F'.

The map $F'\colon \mathscr{N}' \to \mathscr{M}'$ defined by (3.2) will be called the (φ, ψ)-*dual map* of F (often, if no confusion can arise, we shall simply call it the *dual map* of F). Remark that F' is the (φ, ψ)-dual of F if and only if F is the (ψ', φ')-dual of F'. The following remark is essentially due to Evans [17, 18].

LEMMA 3.2. *In the notations of Proposition* 3.1 *let F be completely positive identity preserving and let $V\colon \mathscr{H} \to \mathscr{K}$ be the unique contraction satisfying*

$$Vm\Phi = F(m)\Psi; \qquad m \in \mathscr{M}.$$

Then the linear map $D\colon \mathscr{M} \to B(\mathscr{K})$ defined by

$$D(m) = F(m) - VmV^{+}; \qquad m \in \mathscr{M}$$

is completely positive.

Proof. As in the proof of Proposition 3.1 it will be sufficient to show that for each $k \in \mathbb{N}$; $n_1',\ldots, n_k' \in \mathscr{N}'$; $m_1,\ldots, m_k \in \mathscr{M}$,

$$\sum_{i,j=1}^{k} \langle n_i'\Psi, D(m_i^* m_j)\, n_j'\Psi \geqslant 0$$

and this follows from the equalities

$$\begin{aligned}
&\langle n_i'\Psi, D(m_i^* m_j)\, n_j'\Psi\rangle \\
&\quad = \langle n_i{}'\Psi, F(m_i^* m_j)\, n_j'\Psi\rangle - \langle F'(n_i')\Phi, m_i^* m_j F'(n_j')\Phi\rangle \\
&\quad = \langle F'(n_j'{}^* n_i')\Phi, m_i^* m_j\Phi\rangle - \langle m_i\Phi, F'(n_i')^* F'(n_j')\, m_j\Phi\rangle \\
&\quad = \langle m_i\Phi, [F'(n_i'{}^* n_j') - F'(n_i')^* F(n_j')]\, m_j\Phi\rangle
\end{aligned}$$

and the fact that, F' being completely positive, the quadratic form

$$(\xi_1,\ldots, \xi_k) \in \mathscr{H}^k \mapsto \sum_{i,j=1}^{k} \langle \xi_i, [F'(n_i'{}^* n_j') - F(n_i')^* F'(n_j')]\, \xi_j\rangle$$

is positive.

Now, in the notations introduced in Section 2, let $F\colon \mathfrak{A} \to \mathscr{B}$ be a completely positive, identity preserving, φ-compatible map, and let us introduce the notation

$$V^+ a\Phi = F(a)\Phi; \qquad a \in \mathcal{A}, \tag{3.4}$$

then V^+ extends to a contraction $[\mathcal{A}\Phi] \to [\mathcal{B}\Phi]$ which will be still denoted V^+. In the classical case (i.e., if $\mathcal{A}$ and $\mathcal{B}$ are maximal abelian on the corresponding spaces) the conditional expectation $F: \mathcal{A} \to \mathcal{B}$ associated to φ satisfies, in particular,

$$\varphi_0(F(b_1)\,F(b_2)) = \varphi(b_1 b_2); \qquad b_1, b_2 \in \mathcal{B}$$

and this is equivalent to saying that $V = (V^+)^+$ is a partial isometry with initial projection $[\mathcal{B} \cdot \Phi]$. In the general case we have (cf. [13a] for a similar result):

PROPOSITION (3.3). *For an F as above the following are equivalent*:

(i) *V is a partial isometry with initial projection $[\mathcal{B} \cdot \Phi]$.*

(ii) *The dual map of F is an embedding.*

(iii) *$F(a) = V^+ aV$; $a \in \mathcal{A}$.* (3.5)

(iv) *For any $b \in \mathcal{B}$ there exists a sequence (a_n) in $\mathcal{A}$ such that*:

$$\|a_n\Phi\| \leqslant \|b\Phi\|, \qquad \textit{for each } n \in \mathbb{N}, \tag{3.6}$$

$$\lim_n F(a_n)\Phi = b\Phi. \tag{3.7}$$

Proof. (i) $\Leftrightarrow$ (iii). In the notations of Lemma 3.2, condition (i) is equivalent to $D(1_{\mathcal{A}}) = F(1_{\mathcal{A}}) - V^+ 1_{\mathcal{A}} V = 1_{\mathcal{B}} - V^+V = 0$. Since $D(\cdot)$ completely positive, this is equivalent to $D \equiv 0$, which is (iii).

(iii) $\Leftrightarrow$ (ii). Condition (iii) is equivalent to:

$$\langle b_1'\Phi, F(a)\, b_2'\Phi\rangle = \langle b_1'\Phi, V^+ aVb_2'\Phi_2\rangle$$

for any $b_1, b_2 \in \mathcal{B}$, $a \in \mathcal{A}$; and since, by definition of dual map

$$\langle b_1'\Phi, F(a)\, b_2'\Phi\rangle = \langle F'(b_2'^* b_1')\Phi, a\Phi\rangle,$$

$$\langle b_1'\Phi, V^+ aVb_2'\Phi\rangle = \langle F'(b_2')^* F'(b_1')\Phi, a\Phi\rangle,$$

this is equivalent to $F'(b_1'^* b_1') = F'(b_2'^*)\, F'(b_2')$, which is (ii). The equivalence between (i) and (iv) is a consequence of the following lemma.

LEMMA (3.4). *For any contraction V: $[\mathcal{A}\Phi] \to]\mathcal{B}\Phi]$ the following are equivalent*:

(j) $VV^+ = P_{[\mathscr{B}\Phi]}$.

(jj) *$V\mathcal{A}\Phi \subseteq [\mathscr{B}\Phi]$ and for every $b \in \mathscr{B}$ there exists a sequence (a_n) in $\mathcal{A}$ such that*

$$\|a_n \Phi\| \leqslant \|b\Phi\|, \tag{3.8}$$

$$\lim_n Va_n \Phi = b\Phi. \tag{3.9}$$

The implication (j) $\Rightarrow$ (jj) is clear since, for any $b \in \mathscr{B}$, $V^+ b\Phi \in [\mathcal{A}\Phi]$ hence there is a sequence (a_n) in $\mathcal{A}$ such that $\lim a_n \Phi = V^+ b\Phi$ and there are no problems in choosing (a_n) so that (3.8) is satisfied. Equation (3.9) follows then by continuity. To prove the converse implication, let $b \in \mathscr{B}$ and let (a_n) be a sequence in $\mathcal{A}$ satisfying (3.8) and (3.9). Then we can assume, possibly substituting (a_n) with a sub-sequence, that there is a $\xi \in [\mathcal{A} \cdot \Phi]$ which is the weak limit of the sequence $(a_n \Phi)$. From this and (3.9) we deduce:

$$V\xi = w\text{-lim } Va_n \Phi = b\Phi.$$

Since V is a contraction

$$\|\xi\| \geqslant \|V\xi\| = \|b\Phi\|$$

and, since $(a_n \Phi)$ converges weakly to ξ, $\|\xi\| \leqslant \liminf \|a_n \Phi\| \leqslant \|b\Phi\|$. From this we conclude that, for any $b \in \mathscr{B}$, there exists a $\xi \in [\mathcal{A}\Phi]$ such that

$$V\xi = b\Phi; \qquad \|\xi\| = \|b\Phi\|. \tag{3.10}$$

Now for any $\zeta_0 \in [\mathscr{B} \cdot \Phi]$ there is a sequence (b_n) in $\mathscr{B}$ such that $(b_n \Phi)$ converges to ζ_0. Choose, for each n, a vector $\zeta_n \in [\mathcal{A} \cdot \Phi]$ which satisfies conditions (3.10) for b_n. The sequence (ζ_n) is bounded, hence we can assume that it has a weak limit $\zeta \in [\mathcal{A} \cdot \Phi]$ (by choosing, in case, a sub-sequence). In particular

$$V\zeta = w\text{-lim } V\zeta_n = w\text{-lim } b_n \Phi = \zeta_0$$

and $\|\zeta\| = \|\zeta_0\|$, since $\|\zeta\| \geqslant \|V\zeta\| = \|\zeta_0\|$ and

$$\|\zeta\| \leqslant \text{lim-inf} \|\zeta_n\| = \text{lim-inf} \|b_n \Phi\| = \|\zeta_0\|.$$

Thus for any $\zeta_0 \in [\mathscr{B} \cdot \Phi]$ there is a $\zeta \in [\mathcal{A} \cdot \Phi]$ such that

$$V\zeta = \zeta_0; \qquad \|\zeta\| = \|\zeta_0\|. \tag{3.11}$$

Because of our assumptions this implies that $[\mathscr{B} \cdot \Phi]$ is the range of V. Now, the set

$$K' = \{\zeta \in \mathscr{H} : \|V\zeta\| = \|\zeta\|\}$$

is clearly closed and closed under scalar multiplication. Moreover, since for $\zeta_1, \zeta_2 \in K'$

$$\|\zeta_1\|^2 + \|\zeta_2\|^2 + 2\,\mathrm{Re}\langle V\zeta_1, V\zeta_2\rangle$$
$$= \|V(\zeta_1 + \zeta_2)\|^2 \leqslant \|\zeta_1 + \zeta_2\|^2$$

we have

$$\mathrm{Re}\langle V\zeta_1, V\xi_2\rangle \leqslant \langle \zeta_1, \zeta_2\rangle$$

which, since K' is closed under scalar multiplication, is equivalent to

$$\langle V\zeta_1, V\zeta_2\rangle = \langle \zeta_1, \zeta_2\rangle; \qquad \zeta_1, \zeta_2 \in K'.$$

From this one easily deduces that K' is a closed linear space. To prove that V is a partial isometry with final projection $[\mathscr{B}\cdot\Phi]$ (and initial projection K') it will be sufficient to prove that $V\zeta = 0$ if ζ is orthogonal to K'. But, given a ζ, $V\zeta \in [\mathscr{B}\cdot\Phi] =$ range of V, hence by (3.11) there is a $\zeta_1 \in \mathscr{H}$ such that

$$V\zeta_1 = V\zeta; \qquad \|V\zeta\| = \|\zeta_1\|. \tag{3.12}$$

Hence, for any $\alpha > 0$

$$\|V(\alpha\zeta + \zeta_1)\|^2 = (\alpha + 1)^2\,\|V\zeta\|^2 \tag{3.13}$$

and, since $\zeta_1 \in K'$,

$$\|\alpha\zeta + \zeta_1\|^2 = \alpha^2\,\|\zeta\|^2 + \|V\zeta\|^2. \tag{3.14}$$

Since V is a contraction, (3.13) and (3.14) imply

$$(1 + 2/\alpha)\,\|V\zeta\|^2 \leqslant \|\zeta\|^2$$

which can hold for arbitrary $\alpha > 0$ only if $V\zeta = 0$. Therefore $VV^+ = P_{[\mathscr{B}\cdot\Phi]}$, and this ends the proof.

Remark. In Lemma 3.4 and Proposition 3.3 we used the assumption $\mathscr{B} \subseteq \mathscr{A}$ only to make clear our motivation to focus on partial isometries. One easily sees, by inspection of the proof, that these assertions hold in the more general assumptions of Proposition 3.1.

Now, as remarked in the Introduction, the Tomita involutions set up a one-to-one correspondence between the φ'-compatible embeddings $u': \mathscr{B}' \hookrightarrow \mathscr{A}'$ and the φ-compatible embeddings $u: \mathscr{B} \hookrightarrow \mathscr{A}$, namely,

$$u' = j_{\mathscr{A}} u j_{\mathscr{A}}^{-1}. \tag{3.15}$$

If u' is the φ-dual of a map $F: \mathcal{A} \to \mathcal{B}$ then u, defined by (3.15), will be called the *φ-bidual map* of F (and F the φ-bidual map of u). In the classical case $\mathcal{A}' = \mathcal{A}$, $\mathcal{B}'\mathcal{B}$, and $j_{\mathcal{A}}$ coincides with the algebraic involution, hence, when restricted on $\mathcal{B}$, with $j_{\mathcal{B}}$. Thus in (3.15), $u' = u$ and, from (3.2) we recognize that the classical conditional expectation associated to φ is characterized by the property of being the (unique) φ-bidual of the identity embedding $\imath: \mathcal{B} \hookrightarrow \mathcal{A}$. In the general case the φ-bidual of the identity embedding $\imath: \mathcal{B} \hookrightarrow \mathcal{A}$ is still defined as the φ'-dual of the embedding:

$$u' = j_{\mathcal{A}} j_{\mathcal{B}}^{-1}: \mathcal{B}' \hookrightarrow \mathcal{A}'. \tag{3.16}$$

Clearly u' is φ'-compatible, normal, faithful, identity preserving and completely positive. Therefore, according to Proposition 3.1, its dual map $E: \mathcal{A} \to \mathcal{B}$, characterized by

$$\langle u'(b')\Phi, a\Phi \rangle = \langle b'\Phi, E(a)\Phi \rangle \tag{3.17}$$

for all $b' \in \mathcal{B}'$, $a \in \mathcal{A}$, is φ-compatible and enjoys all these properties. Since u' is an embedding then, according to Proposition 3.3' E has the form

$$E(a) = U^+ a U, \qquad a \in \mathcal{A}, \tag{3.18}$$

where U: $[\mathcal{A} \cdot \Phi] \to [\mathcal{B} \cdot \Phi]$ is the partial isometry with initial projection $[\mathcal{B} \cdot \Phi] = [\mathcal{B}' \cdot \Phi]$, characterized by

$$U b' \Phi = u(b')\Phi, \qquad b' \in \mathcal{B}' \tag{3.19}$$

or, equivalently

$$U = J_{\mathcal{A}} J_{\mathcal{B}} P_{[\mathcal{B} \cdot \Phi]} \tag{3.20}$$

(cf. Sect. 2 for the notations). While if $F: \mathcal{A} \to \mathcal{B}$ is the φ-bidual of a generic φ-compatible embedding $v: \mathcal{B} \hookrightarrow \mathcal{A}$, one has

$$F(a) = J_{\mathcal{B}} V_0^+ J_{\mathcal{A}} a J_{\mathcal{A}} V_0 J_{\mathcal{B}} = V^+ a V; \qquad a \in \mathcal{A}, \tag{3.21}$$

where V_0 is the partial isometry in $\mathcal{H}$ with initial projection $[\mathcal{B}\Phi]$ characterized by

$$V_0 b \Phi = v(b)\Phi, \qquad b \in \mathcal{B}, \tag{3.22}$$

$$v' = j_{\mathcal{A}} v j_{\mathcal{B}}^{-1} \qquad \text{and} \qquad V = J_{\mathcal{A}} V_0 J_{\mathcal{B}}. \tag{3.23}$$

In the following we will frequently use these notations. We sum up our discussion in the following

THEOREM 3.5. *Let $\mathcal{A}$ be a von Neumann algebra and φ a faithful normal state on $\mathcal{A}$. For any sub-algebra $\mathcal{B} \subseteq \mathcal{A}$, there exists a map $E: \mathcal{A} \to \mathcal{B}$ characterized by the following properties:*

E is completely positive identity preserving, faithful, normal. (CE0)

$$\varphi_0 \cdot E = \varphi. \tag{CE1}$$

The map induced by E on the GNS *space of $\{\mathcal{A}, \varphi\}$ is a partial isometry with final projection $[\mathcal{B} \cdot \Phi]$.* (CE2)

The φ-bidual of E is the identity embedding $\iota: \mathcal{B} \hookrightarrow \mathcal{A}$. (CE3)

The explicit form of E is given by (3.18), (3.20).

Proof. From the consideration above.

Remark. As already remarked, condition (CE3) alone is sufficient to characterize the map E.

In the classical case E coincides with the usual conditional expectation defined by φ; when the conditions of Takesaki's theorem [31] are fulfilled, E coincides with the conditional expectation defined by that theorem (cf. Theorem 5.2); in the case of matrix algebras the explicit form of E coincides with the expression sugested in [1] (cf. also [2]) on the ground of probabilistic considerations, and in the case of semi-finite algebras it gives a rigorous meaning to the heuristic extrapolation of this expression (cf. formulae (3.28), (3.29)). For these reasons we call E the *φ-conditional expectation from $\mathcal{A}$ into $\mathcal{B}$*. In the rest of the paper the symbols u', E, U will denote the maps defined respectively by (3.16), (3.18), (3.20). The use of the term "φ-conditional expectation" to denote a map characterized by (CE0)–(CE3) (which in general is not a projection) cannot create confusion with the term "conditional expectation" currently used in the literature to denote a norm one projection. In fact the former expression makes sense only when it refers to a given state φ and, as already remarked, when a norm one projection from $\mathcal{A}$ into $\mathcal{B}$ satisfying (CE0)–(CE3) exists, it must coincide with the φ-conditional expectation.

Remark. In the notations (3.21)–(3.23), we have, for $a \in \mathcal{A}$:

$$F(a) = J_{\mathcal{B}} V_0^+ J_{\mathcal{A}} a J_{\mathcal{A}} V_0 J_{\mathcal{B}} = U^+ (V_1 a V_1) U$$

with $V_1 = J_{\mathcal{A}} V J_{\mathcal{A}}$. Thus formally $F(a) = E(V_1^+ a V_1)$, however, this is in general only a formal expression, since $V_1^+ \mathcal{A} V_1$ in general is not in $\mathcal{A}$. More insight into the structure of the φ-conditional expectation is given by the following:

PROPOSITION 3.6. In the notations and the assumptions (3.21)–(3.23) the following equalities hold:

$$\Delta_{\mathscr{B}}^{-1/2} V_0^+ \Delta_{\mathfrak{A}}^{1/2} = J_{\mathscr{B}} V_0^+ J_{\mathfrak{A}} = V^+, \tag{3.24}$$

$$\Delta_{\mathfrak{A}}^{1/2} V_0 \Delta_{\mathscr{B}}^{-1/2} P = J_{\mathfrak{A}} V_0 J_{\mathscr{B}} P = V \tag{3.25}$$

in the sense that the left hand sides are well defined in appropriate dense domains (specified in the proof) and on these domains the equalities hold.

Proof. Let us first prove the (equivalent) equalities

$$\Delta_{\mathscr{B}}^{1/2} V^+ \Delta_{\mathfrak{A}}^{-1/2} = J_{\mathscr{B}} V^+ J_{\mathfrak{A}} = V_0^+, \tag{3.26}$$

$$\Delta_{\mathfrak{A}}^{-1/2} V \Delta_{\mathscr{B}}^{1/2} P = J_{\mathfrak{A}} V J_{\mathscr{B}} P = V_0. \tag{3.27}$$

Let $a' \in \mathfrak{A}'$, then $a'\Phi \in \mathscr{D}(\Delta_{\mathfrak{A}}^{-1/2})$; and $\Delta_{\mathfrak{A}}^{-1/2} a'\Phi = j_{\mathfrak{A}}(a')^*\Phi \in \mathfrak{A} \cdot \Phi$. Hence

$$V^+ \Delta_{\mathfrak{A}}^{-1/2} a'\Phi = V^+ j_{\mathfrak{A}}(a')^*\Phi = F(j_{\mathfrak{A}}(a')^*)\Phi \in \mathscr{B} \cdot \Phi \subseteq \mathscr{D}(\Delta_{\mathscr{B}}^{1/2}),$$

$$\Delta_{\mathscr{B}}^{1/2} V^+ \Delta_{\mathfrak{A}}^{-1/2} a'\Phi = J_{\mathscr{B}} F(j_{\mathfrak{A}}(a'))\Phi = J_{\mathscr{B}} V^+ J_{\mathfrak{A}} V^+ J_{\mathfrak{A}} a'\Phi.$$

Therefore $\Delta_{\mathscr{B}}^{1/2} V^+ \Delta_{\mathfrak{A}}^{-1/2}$ can be extended to a bounded operator on $\mathscr{H}$ (denoted with the same symbol) and (3.26) holds. Similarly, if $b \in \mathscr{B}$, then $\Delta_{\mathscr{B}}^{1/2} b\Phi = j_{\mathscr{B}}(b^*)\Phi \in \mathscr{B} \cdot \Phi$, hence

$$V\Delta_{\mathscr{B}}^{1/2} b\Phi = v'(j(b^*))\Phi \in \mathfrak{A}'\Phi \subseteq \mathscr{D}(\Delta_{\mathfrak{A}}^{-1/2})$$

$$\Delta_{\mathfrak{A}}^{-1/2} V \Delta_{\mathscr{B}}^{1/2} b\Phi = J_{\mathfrak{A}} F_{\mathfrak{A}} v'(j_{\mathscr{B}}(b))^*\Phi = J_{\mathfrak{A}} V J_{\mathscr{B}} b\Phi$$

and from this (3.27) follows.

Since $V = J_{\mathfrak{A}} V_0 J_{\mathscr{B}} P$, (3.26) and (3.27) are equivalent respectively to

$$\Delta_{\mathscr{B}}^{-1/2} V_0^+ \Delta_{\mathfrak{A}}^{1/2} J_{\mathfrak{A}} = J_{\mathscr{B}} V_0^+,$$

$$\Delta_{\mathfrak{A}}^{1/2} V_0 \Delta_{\mathscr{B}}^{-1/2} J_{\mathscr{B}} P = J_{\mathfrak{A}} V_0,$$

which are easily seen to be equivalent to (3.24), (3.25), respectively.

Remark 1. In particular, for te φ-conditional expectation one has

$$\Delta_{\mathscr{B}}^{-1/2} P \Delta_{\mathfrak{A}}^{1/2} = U^+, \qquad \text{on } \mathfrak{A} \cdot \Phi; \tag{3.28}$$

$$\Delta_{\mathfrak{A}}^{1/2} \Delta_{\mathscr{B}}^{-1/2} P = U, \qquad \text{on } \mathscr{B}' \cdot \Phi. \tag{3.29}$$

Remark 2. A useful substitute for the projection property of the classical φ-conditional expectations is the *chain rule* satisfied in the general case. Namely, if $\mathscr{C} \subseteq \mathscr{B} \subseteq \mathfrak{A}$ are three W^*-algebras, φ is a faithful state on $\mathfrak{A}$ and

$E_{\mathcal{A},\mathcal{B}}$, $E_{\mathcal{B},\mathcal{C}}$, $E_{\mathcal{A},\mathcal{C}}$ are the φ-conditional expectations defined for the couples $\mathcal{B} \subseteq \mathcal{A}$, $\mathcal{C} \subseteq \mathcal{B}$, $\mathcal{C} \subseteq \mathcal{A}$, respectively, then

$$E_{\mathcal{B},\mathcal{C}} \cdot E_{\mathcal{A},\mathcal{B}} = E_{\mathcal{A},\mathcal{C}}. \tag{3.30}$$

More generally, if $v_{\mathcal{C},\mathcal{B}}: \mathcal{C} \hookrightarrow \mathcal{B}$ and $v_{\mathcal{B},\mathcal{A}}: \mathcal{B} \hookrightarrow \mathcal{A}$ are φ-compatible embeddings and $F_{\mathcal{A},\mathcal{B}}$, $F_{\mathcal{B},\mathcal{C}}$ their biduals one has for each $a \in \mathcal{A}$

$$\begin{aligned} F_{\mathcal{B},\mathcal{C}} F_{\mathcal{A},\mathcal{B}}(a)\Phi &= J_{\mathcal{C}} V^+_{\mathcal{C},\mathcal{B}} J_{\mathcal{B}} \cdot J_{\mathcal{B}} V^+_{\mathcal{B},\mathcal{A}} J_{\mathcal{A}} a\Phi \\ &= J_{\mathcal{C}} V^+_{\mathcal{C},\mathcal{B}} V^+_{\mathcal{B},\mathcal{A}} J_{\mathcal{A}} a\Phi \end{aligned}$$

(here $V_{\mathcal{C},\mathcal{B}}, \ldots$ play the role of V_0 in Proposition 3.6). But $V_{\mathcal{B},\mathcal{A}} V_{\mathcal{C},\mathcal{B}}$ is a partial isometry with initial projection $[\mathcal{C} \cdot \Phi]$ and, for $c \in \mathcal{C}$

$$V_{\mathcal{B},\mathcal{A}} V_{\mathcal{C},\mathcal{B}} c\Phi = v_{\mathcal{B},\mathcal{A}} v_{\mathcal{C},\mathcal{B}}(c)\Phi$$

thus $F_{\mathcal{B},\mathcal{C}} \cdot F_{\mathcal{A},\mathcal{B}}$ is the bidual map of $v_{\mathcal{B},\mathcal{A}} v_{\mathcal{C},\mathcal{B}}$ and this, in particular, implies (3.34).

Finally, let us remark that the consideration of bidual maps is a necessary step, in the sense that without it we cannot go much besides the conditions imposed by Takesaki's theorem. In fact one has

PROPOSITION (3.7). *Let $v: \mathcal{B} \hookrightarrow \mathcal{A}$ be a φ-compatible embedding and let V be the partial isometry with initial projection $[\mathcal{B} \cdot \Phi]$ characterized by*

$$Vb\Phi = u(b)\Phi; \qquad b \in \mathcal{B}. \tag{3.31}$$

The following are equivalent:

(i) $V^+ \mathcal{A} V \subseteq \mathcal{B}$,

(ii) *there exists a faithful normal φ-compatible norm one projection F of $\mathcal{A}$ onto $v(\mathcal{B})$.*

Proof. (i) $\Rightarrow$ (ii). Since v is an embedding of $\mathcal{B}$ into $\mathcal{A}$ one easily verifies that $Vb = v(b)V$ ($b \in \mathcal{B}$) or, equivalently

$$bP = V^+ v(b) V; \qquad b \in \mathcal{B}. \tag{3.32}$$

Since $v(\mathcal{B}) \subseteq \mathcal{A}$, (3.32) and (i) imply that $V^+ \mathcal{A} V = \mathcal{B}$. Define $F_1 : \mathcal{A} \to \mathcal{B}$ by $F_1(a) = V^+ a V$. Then F_1 is normal, onto and, because of (3.32), $F = vF_1 : \mathcal{A} \to v(\mathcal{B})$ is a norm one normal projection onto $v(\mathcal{B})$. Clearly F is φ-compatible hence faithful. (ii) $\Rightarrow$ (i). Let F be as in (ii). Denoting

$v^{-1}: v(\mathscr{B}) \to \mathscr{B}$ the left inverse of v, and $G = v^{-1}F: \mathfrak{A} \to \mathscr{B}$, one has for each $a \in \mathfrak{A}$ and $b \in \mathscr{B}$:

$$VG(a)\,b\Phi = v(G(a))\,vb\Phi = F(a)\,v(b)\Phi$$
$$= F(av(b))\Phi = P_{[v(\mathscr{B})\cdot\Phi]}aVb\Phi$$

or, equivalently,

$$G(a)\,b\Phi = V^{+}aVb\Phi,$$

which implies (i).

4. The Range of the φ-Conditional Expectation

If F is the bidual of a φ-compatible embedding $v: \mathscr{B} \hookrightarrow \mathfrak{A}$ its range is nontrivial since $F(\mathfrak{A})\Phi$, being the image of $\mathfrak{A}\Phi$ under a partial isometry with final projection $[\mathscr{B} \cdot \Phi]$, is dense in $[\mathscr{B} \cdot \Phi]$. More precisely, we have the following, which is a remnant of the property $F(a \cdot b) = F(a) \cdot b$ $(a \in \mathfrak{A}, b \in \mathscr{B})$ characterizing the classical conditional expectations:

Proposition 4.1. *Let F be as above and let V be the partial isometry with initial projection $[\mathscr{B}\Phi]$ characterized by*

$$V^{+}a\Phi = F(a)\Phi; \qquad a \in \mathfrak{A}. \tag{4.1}$$

Then:

(i) *For each $b \in \mathscr{B}$ there is a sequence (a_n) in $\mathfrak{A}$ such that for $n \to \infty$*

$$a_n\Phi \to Vb\Phi; \qquad F(a_n)\Phi \to b\Phi; \qquad \|a_n\Phi\| \leqslant \|b\Phi\| \tag{4.2}$$

$$F(a \cdot a_n)\Phi \to F(a)\,b\Phi;\ a \in \mathfrak{A}; \tag{4.3}$$

(ii) *$\varphi \restriction \mathscr{B}$ is uniquely determined by its restriction on $F(\mathfrak{A})$.*

Proof. Let $b \in \mathscr{B}$. The existence of a sequence (a_n) in $\mathfrak{A}$ satisfying (4.2) has been established in the proof of the implication (j) $\Rightarrow$ (jj) in Lemma 3.4. For such a sequence one has

$$\|F(a \cdot a_n)\Phi - F(a)\,b\Phi\|$$
$$= \|V^{+}aa_n\Phi - V^{+}aVb\Phi\|$$
$$\leqslant \|a\| \cdot \|a_n\Phi - Vb\Phi\| \to 0$$

which proves (4.3). Taking $a = 1$ in (4.3) we see that $\varphi \restriction \mathscr{B}$ is uniquely determined by its restriction on $F(\mathfrak{A})$, and this proves (i) and (ii).

The results of Proposition 4.1 show that the range of the φ-conditional expectation or of any bidual of a φ-compatible embedding is large in $\mathscr{B}$ with respect to φ (we will say φ-dense). However, in general the φ-conditional expectation *is not surjective* as the following example shows: assume that $\mathcal{O}\!\!\!\!\mathcal{C} \neq \mathscr{B}$ and that Φ is cyclic and separating in $\mathscr{H}$ for both $\mathcal{O}\!\!\!\!\mathcal{C}$ and $\mathscr{B}$. Then in the notations (3.21)–(3.23), we have that $F(\mathcal{O}\!\!\!\!\mathcal{C}) = \mathscr{B}$ if and only if

$$F(\mathcal{O}\!\!\!\!\mathcal{C}) = J_{\mathscr{B}} V_0^+ J_{\mathcal{O}\!\!\!\!\mathcal{C}}(\mathcal{O}\!\!\!\!\mathcal{C})\, J_{\mathcal{O}\!\!\!\!\mathcal{C}} V_0 J_{\mathscr{B}} = \mathscr{B}. \tag{4.4}$$

But in this case V_0 is unitary, therefore (4.4) is equivalent to $V_0^+ \mathcal{O}\!\!\!\!\mathcal{C} V_0 = \mathscr{B}$. Thus $\alpha: a \in \mathcal{O}\!\!\!\!\mathcal{C} \to \alpha(a) = V_0^+ a V_0 \in \mathscr{B}$ is a φ-compatible endomorphism of $\mathcal{O}\!\!\!\!\mathcal{C}$ hence it commutes with the modular automorphism group $\sigma^t_{\mathcal{O}\!\!\!\!\mathcal{C}}$ of $\mathcal{O}\!\!\!\!\mathcal{C}$ associated with φ (cf. [7, 29] and Proposition 6.1 in the present paper). Therefore it (4.4) holds:

$$\sigma^t_{\mathcal{O}\!\!\!\!\mathcal{C}}(\mathscr{B}) = \sigma^t_{\mathcal{O}\!\!\!\!\mathcal{C}}(\alpha(\mathcal{O}\!\!\!\!\mathcal{C})) = \alpha\sigma^t_{\mathcal{O}\!\!\!\!\mathcal{C}}(\mathcal{O}\!\!\!\!\mathcal{C}) = \alpha(\mathcal{O}\!\!\!\!\mathcal{C}) = \mathscr{B}$$

hence, by Takesaki's theorem (cf. [31] and Theorem 5.2 in the present paper) the φ-conditional expectation is a norm one projection of $\mathcal{O}\!\!\!\!\mathcal{C}$ onto $\mathscr{B}$ thus

$$J_{\mathscr{B}} J_{\mathcal{O}\!\!\!\!\mathcal{C}}(\mathcal{O}\!\!\!\!\mathcal{C})\, J_{\mathcal{O}\!\!\!\!\mathcal{C}} J_{\mathscr{B}} = \mathscr{B} \Leftrightarrow \mathcal{O}\!\!\!\!\mathcal{C} = \mathscr{B}$$

against the assumption $\mathcal{O}\!\!\!\!\mathcal{C} \neq \mathscr{B}$. Therefore, in our assumptions one must have

$$\mathcal{O}\!\!\!\!\mathcal{C} \supset \mathscr{B} \supset F(\mathcal{O}\!\!\!\!\mathcal{C}) \tag{4.5}$$

($\supset$ means *strict* inclusion). In particular, for the φ-conditional expectation we can define for each $n \in \mathbb{Z}$

$$\mathcal{O}\!\!\!\!\mathcal{C}_n = (U^+)^n \mathcal{O}\!\!\!\!\mathcal{C} U^n; \qquad \mathscr{B}_n = (U^+)^n \mathscr{B} U^n; \qquad U = J_{\mathcal{O}\!\!\!\!\mathcal{C}} J_{\mathscr{B}}$$

and denote $\varphi_n = \varphi \upharpoonright \mathcal{O}\!\!\!\!\mathcal{C}_n$. From (4.5)—with $F = E$—one immediately deduces that

$$\mathcal{O}\!\!\!\!\mathcal{C}_n \supset \mathscr{B}_n \supset \mathcal{O}\!\!\!\!\mathcal{C}_{n+1}; \tag{4.6}$$

and clearly Φ is cyclic and separating in $\mathscr{H}$ for each couple $\{\mathcal{O}\!\!\!\!\mathcal{C}_n, \mathscr{B}_n\}$.

Now, denoting $J_{\mathcal{O}\!\!\!\!\mathcal{C}_n}$, $J_{\mathscr{B}_n}$ the Tomita involutions associated to $\mathcal{O}\!\!\!\!\mathcal{C}_n$, $\mathscr{B}_n$, respectively, with respect to the state φ_n, one has

$$J_{\mathcal{O}\!\!\!\!\mathcal{C}_n} = (U^+)^n J_{\mathcal{O}\!\!\!\!\mathcal{C}} U^n; \qquad J_{\mathscr{B}_n} = (U^+)^n J_{\mathscr{B}} U^n$$

therefore $J_{\mathcal{O}\!\!\!\!\mathcal{C}_n} \cdot J_{\mathscr{B}_n} = (U^+)^n (J_{\mathcal{O}\!\!\!\!\mathcal{C}} J_{\mathscr{B}})\, U^n = (U^+)^n U^{n+1} = U$. Hence if $a_n \in \mathcal{O}\!\!\!\!\mathcal{C}_n$:

$$J_{\mathscr{B}_n} J_{\mathcal{O}\!\!\!\!\mathcal{C}_n}(a_n)\, J_{\mathcal{O}\!\!\!\!\mathcal{C}n} J_{\mathscr{B}_n} = U^+ a_n U$$

thus if $n \geqslant 0$ we have that $E_n = E \upharpoonright \mathcal{A}_n$ is the φ_n-conditional expectation from $\mathcal{A}_n$ into $\mathcal{B}_n$. The construction above provides an examples of a map, namely, $E_n : \mathcal{A}_n \to \mathcal{A}_{n+1}$, identity preserving, φ_n-invariant and surjective but which, for the arguments above, cannot be the bidual of a φ_n-compatible embedding of $\mathscr{A}_{n+1}$ into $\mathcal{A}_n$.

To formulate some algebraic properties of the range of the φ-conditional expectation, let us recall some known facts about completely positive identity preserving maps $F: \mathcal{A} \to \mathcal{A}$ (cf., for example, [17, 18]). Define:

$$\mathscr{R}(F) = \{c \in \mathcal{A}: F(c^* \cdot c) = F(c)^* \cdot F(c)\},$$

$$\mathscr{L}(F) = \{c \in \mathcal{A}: F(cc^*) = F(c) \cdot F(c)^*\},$$

$$\mathcal{A}_F = \{c \in \mathcal{A}: F(c) = c\}.$$

It is known that $c \in \mathscr{R}(F)$ (resp. $\mathscr{L}(F)$ if and only if for each $a \in \mathcal{A}$

$$F(ac) = F(a)\,F(c) \qquad (\text{resp. } F(ca) = F(c)\,F(a))$$

and that $\mathscr{R}(F)$ and $\mathscr{L}(F)$ are algebras, but in general not $*$-algebras, moreover $\mathscr{R}(F)^* = \mathscr{L}(F)$.

If F is φ-compatible one has

$$\mathcal{A}_F \subseteq \mathscr{R}(F) \cap \mathscr{L}(F) \tag{4.7}$$

in fact, if $c \in \mathcal{A}_F$ then from $\varphi(F(c^* \cdot c) - c^* \cdot c) = 0$, and the Kadison–Schwarz inequality [24] we deduce $0 = F(c^* \cdot c) - c^* \cdot c = F(c^* \cdot c) - F(c)^* \cdot F(c)$ i.e., $c \in \mathscr{R}(F)$, and similarly for $\mathscr{L}(F)$. In particular, $\mathcal{A}_F$ is the C^*-subalgebra (W^*—if F is normal) of $\mathcal{A}$:

$$\mathcal{A}_F = \{c \in \mathcal{A}: F(ac) = F(a)c; F(ca) = cF(a); a \in \mathcal{A}\}. \tag{4.8}$$

PROPOSITION 4.2. *In the assumptions and the notations of Proposition* 4.1 *one has*

$$\mathscr{R}(F) = \{c \in \mathcal{A} : \| V^+ c\Phi \| = \|c\Phi\|\}, \tag{4.9}$$

$$\mathscr{R}(F)\Phi = \mathcal{A}\Phi \cap V[\mathscr{B} \cdot \Phi] = \mathcal{A}\Phi \cap [v'(\mathscr{B}') \cdot \Phi], \tag{4.10}$$

$$\mathscr{R}(F) = \{c \in \mathcal{A}: c[v'(\mathscr{B}')\Phi] \subseteq [v'(\mathscr{B}')\Phi] \tag{4.11}$$

$$(v' = j_{\mathcal{A}} v j_{\mathscr{B}}).$$

Proof. Let $c \in \mathcal{A}$, then $\| V^+ c\Phi \| = \|c\Phi\|$ if and only if:

$$0 = \|V^+ c\Phi\|^2 - \|c\Phi\| = \varphi([F(c)^* \cdot F(c) - F(c^*c)])$$

thus, since φ is faithful, (4.9) follows from the Kadison–Schwarz inequality.

Because of (4.6) and the fact that Φ is separating, we have for any $a \in \mathcal{A}$

$$c \in \mathcal{R}(F) \Leftrightarrow F(ac)\Phi = F(a)\,F(c)\Phi,$$
$$\Leftrightarrow V^{+}ac\Phi = V^{+}aVV^{+}c\Phi.$$

This is equivalent to saying that, for every $b' \in \mathcal{B}'$ and $a \in \mathcal{A}$:

$$\langle a^{*}v'(b')\Phi, c\Phi\rangle = \langle a^{*}v'(b')\Phi, Qc\Phi\rangle, \tag{4.12}$$

where $Q = VV^{+}$ is the final projection of V. By cyclicity (4.12) is equivalent to

$$c\Phi = Qc\Phi \Leftrightarrow c\Phi \in V[\mathcal{B}\Phi],$$

and this proves (4.10). To prove (4.11), remark that

$$V^{+}acV = V^{+}aQvV \Leftrightarrow c \in \mathcal{R}(F)$$

and the left hand side equality is equivalent to

$$\langle b'\Phi, Vac\Phi\rangle = \langle b'\Phi, VaQc\Phi\rangle.$$

This equality can be written:

$$\langle a^{*}\Phi, c\cdot v'(b'^{*})\Phi\rangle = \langle a^{*}\Phi, Qcv'(b'^{*})\Phi\rangle.$$

Since $a \in \mathcal{A}$ and $b' \in \mathcal{B}'$ are arbitrary, this is equivalent to:

$$QcQ = cQ,$$

which is (4.11).

PROPOSITION 4.3. *The φ-conditional expectation $E\colon \mathcal{A} \to \mathcal{B}$ satisfies*

$$\mathcal{R}(E) = \{c \in \mathcal{A} : Pj_{\mathcal{A}}(c)\Phi = j_{\mathcal{A}}(c)\Phi\}, \tag{4.13}$$

$$\mathcal{R}(E)\Phi = \mathcal{A}\Phi \cap [j_{\mathcal{A}}(\mathcal{B})\Phi] = \mathcal{A}\Phi \cap [j_{\mathcal{A}}(\mathcal{B}')\Phi], \tag{4.14}$$

$$\mathcal{R}(E) = \{c \in \mathcal{A}\colon j_{\mathcal{A}}(c)[\mathcal{B}\Phi] \subseteq [\mathcal{B}\cdot\Phi]\}. \tag{4.15}$$

Proof. Using (4.3) we have that $c \in \mathcal{R}(E)$ if and only if

$$\|J_{\mathcal{B}}PJ_{\mathcal{A}}c\Phi\| = \|c\Phi\| \Leftrightarrow \|Pj_{\mathcal{A}}(c)\Phi\| = \|j_{\mathcal{A}}(c)\Phi\| \tag{4.16}$$

and this proves (4.13). From (4.16) we deduce also that

$$J_{\mathcal{A}}PJ_{\mathcal{A}}c\Phi = c\Phi \Leftrightarrow c\Phi \in J_{\mathcal{A}}PJ_{\mathcal{A}}\mathcal{H} = [j_{\mathcal{A}}(\mathcal{B})\Phi]$$

and this proves (4.14). As for (4.15), let us remark that, according to (4.11), $c \in \mathscr{R}(E)$ if and only if:

$$c[u(\mathscr{B}')\Phi] \subseteq [u(\mathscr{B}')\Phi] \Leftrightarrow c \cdot J_{\mathcal{O}\!\mathcal{L}}[j_{\mathscr{B}}(\mathscr{B}')\Phi] \subseteq J_{\mathcal{O}\!\mathcal{L}}[j_{\mathscr{B}}(\mathscr{B}')\Phi]$$
$$\Leftrightarrow j_{\mathcal{O}\!\mathcal{L}}(c)[\mathscr{B}\Phi] \subseteq [\mathscr{B}\Phi].$$

5. Fixed Points of the φ-Conditional Expectations

THEOREM 5.1. *For any $b \in \mathscr{B}$ the following assertions are equivalent*:

(i) $b \in \mathcal{O}\!\mathcal{L}_E$ (*i.e.*, $E(b) = b$).

(ii) $\sigma^t_{\mathcal{O}\!\mathcal{L}}(b) \in \mathscr{B}$: $t \in \mathbb{R}$.

(iii) $\sigma^t_{\mathcal{O}\!\mathcal{L}}(b) = \sigma^t_{\mathscr{B}}(b)$; $t \in \mathbb{R}$.

Proof. (ii) $\Rightarrow$ (iii). Let f, g: $\mathbb{C} \to \mathbb{R}$ be $\mathscr{C}^\infty$-functions whose Fourier transform has compact support. The KMS condition for $\sigma^t_{\mathscr{B}}$ and the assumption (ii) on b imply (cf. [22, 29]) that for each $b_1 \in \mathscr{B}$

$$\begin{aligned} &\int_{R^2}\int f(s)\; g(t)\, \varphi(\sigma^s_{\mathcal{O}\!\mathcal{L}}(b)\, \sigma^t_{\mathscr{B}}(b_1))\, ds\, dt \\ &\qquad = \int_{R^2}\int f(s)\; g(t-i)\, \varphi(\sigma^t_{\mathscr{B}}(b_1)\, \sigma^s_{\mathcal{O}\!\mathcal{L}}(b))\, ds\, dt \end{aligned} \tag{5.1}$$

and, using the KMS conditions for $\sigma^t_{\mathcal{O}\!\mathcal{L}}$ we find

$$\begin{aligned} &\int_{R^2}\int f(s-i)\; g(t-i)\, \varphi(\sigma^s_{\mathcal{O}\!\mathcal{L}}(b)\, \sigma^t_{\mathscr{B}}(b_1))\, ds\, dt \\ &\qquad = \int_{R^2}\int f(s)\; g(t)\, \varphi(\sigma^s_{\mathcal{O}\!\mathcal{L}}(b)\, \sigma^t_{\mathscr{B}}(b_1))\, ds\, dt. \end{aligned} \tag{5.2}$$

By an argument due to Sirugue and Winnink (cf. [29, Appendix]) equality (5.2) for arbitrary f, g, with the above quoted properties, implies $\varphi(\sigma^s_{\mathcal{O}\!\mathcal{L}}(b)\, \sigma^t_{\mathscr{B}}(b_1))$ depends only on $t - s$. Thus, in particular

$$\varphi(\sigma^t_{\mathcal{O}\!\mathcal{L}}(b)\, \sigma^t_{\mathscr{B}}(b_1)) = \varphi(bb_1). \tag{5.3}$$

But condition (ii) implies that for each $t \in \mathbb{R}$, $\Delta^{it}_{\mathcal{O}\!\mathcal{L}} b\Phi \in [\mathscr{B}\Phi]$, therefore (5.3) is equivalent to the condition

$$\Delta^{-it}_{\mathscr{B}}(\Delta^{it}_{\mathcal{O}\!\mathcal{L}} b\Phi) = b\Phi \Leftrightarrow \Delta^{it}_{\mathcal{O}\!\mathcal{L}} b\Phi = \Delta^{it}_{\mathscr{B}}\, b\Phi \tag{5.4}$$

($t \in R$), and this is equivalent to (iii).

(iii) $\Rightarrow$ (i). As stated above, condition (iii) is equivalent to (5.4) which, by analytic continuation, yields

$$\Delta_{\mathcal{A}}^{1/2} b\Phi = \Delta_{\mathcal{B}}^{1/2} b\Phi \tag{5.5}$$

since $b\Phi \in \mathscr{D}(\Delta_{\mathcal{A}}^{1/2}) \cap \mathscr{D}(\Delta_{\mathcal{B}}^{1/2})$. Multiplying both sides of (5.5) by $J_{\mathcal{B}} \cdot P$ one finds:

$$J_{\mathcal{B}} P \Delta_{\mathcal{A}}^{1/2} b\Phi = b^* \Phi$$

and since this is in $\mathscr{D}(S_{\mathcal{B}})$ we can multiply both sides by $S_{\mathcal{B}}$, obtaining

$$b\Phi = S_{\mathcal{B}} J_{\mathcal{B}} P \,\Delta_{\mathcal{A}}^{1/2} b\Phi = \Delta_{\mathcal{B}}^{-1/2} P \,\Delta_{\mathcal{A}}^{1/2} b\Phi = U^+ b\Phi, \tag{5.6}$$

where, in the last equality, we have used equality (3.28). And (5.6) is equivalent to (i) because Φ is separating and $U^+ b\Phi = E(b)\Phi$.

(i) $\Rightarrow$ (ii). By the L^2-ergodic theorem on completely positive identity preserving maps on von Neumann algebras, which leave a faithful state invariant (cf., for example, [19], where the case of a semigroup rather than a single map is considered) there exists a normal faithful norm one projection $F: \mathcal{A} \to \mathcal{A}_E$ which is onto and φ-compatible (i.e., $\varphi \cdot F = \varphi$). The orthogonal projection $Q: \mathscr{H} \to [\mathcal{A}_E \cdot \Phi]$ is then characterized by

$$Qa\Phi = F(a)\Phi; \qquad a \in \mathcal{A}.$$

Therefore $S_{\mathcal{A}} Q = Q S_{\mathcal{A}}$ hence, being Q bounded (cf. [16, XII.1.5, Lemma 6]), $QF_{\mathcal{A}} = F_{\mathcal{A}} Q$. Thus the unitary operator $(1 - 2Q)$ commutes with $S_{\mathcal{A}}$ and $F_{\mathcal{A}}$, hence also with $|S_{\mathcal{A}}|^2 = \Delta_{\mathcal{A}}$ and with all its powers. In particular $\Delta_{\mathcal{A}}^{it} Q = Q\,\Delta_{\mathcal{A}}^{it}$, and therefore:

$$\begin{aligned}\Delta_{\mathcal{A}}^{it}(\mathcal{A}_E \Phi) &= \Delta_{\mathcal{A}}^{it} F(\mathcal{A})\Phi = \Delta_{\mathcal{A}}^{it} Q\mathcal{A}\Phi = Q\Delta_{\mathcal{A}}^{it}\mathcal{A}\Phi \\ &= Q\mathcal{A}\Phi = \mathcal{A}_E \Phi \subseteq \mathcal{B}\Phi,\end{aligned}$$

that is: $\sigma_{\mathcal{A}}^t(\mathcal{A}_E) \subseteq \mathcal{B}$. Thus (i) $\Rightarrow$ (ii) and this ends the proof.

From the above result one can easily deduce the theorem of Takesaki mentioned in the Introduction, namely,

THEOREM 5.2 (Takesaki [31]). *The following statements are equivalent*:

(i) *$\mathcal{B}$ is globally invariant under the modular automorphis group $\sigma_{\mathcal{A}}^t$ associated with φ.*

(ii) *The φ-conditional expectation E is a norm one projection from $\mathcal{A}$ onto $\mathcal{B}$.*

Proof. The statement that E is a norm one projection (that is, (ii)) is equivalent to: $\mathcal{O}\!\mathcal{l}_E = \mathscr{B}$. And, because of Theorem 5.1, this is equivalent to $\sigma_{\alpha}^{t}(\mathscr{B}) \subseteq \mathscr{B}$.

6. Transition Cocycles

In the following we shall need a variant, due to Frigerio, of a known result (cf. [7, 25, 29]):

PROPOSITION 6.1. *Let $\mathscr{M}$, $\mathscr{N}$ be von Neumann algebras acting respectively on the Hilbert spaces H, K with cyclic and separating Φ, Ψ; let $F\colon \mathscr{M} \to \mathscr{N}$ be a completely positive, identity preserving normal map such that*

$$\varphi = \psi \cdot F; \qquad \psi = \langle \Psi' \cdot \Psi \rangle; \qquad \varphi = \langle \Phi' \cdot \Phi \rangle. \tag{6.1}$$

Then the following are equivalent:

(i) $F\sigma_{\varphi}^{t} = \sigma_{\psi}^{t} F$; $t \in \mathbb{R}$.

(ii) *There exists a unique normal, completely positive, identity preserving map $v\colon \mathscr{N} \to \mathscr{M}$ such that*

$$\psi(nF(m)) = \varphi(v(n)m); \qquad n \in \mathscr{N}; \qquad m \in \mathscr{M}. \tag{6.2}$$

Proof. Denoting $V\colon H \to K$ the contraction defined by $Vm\Phi = F(m)\Psi$; $(m \in \mathscr{M})$, it is known [25] that condition (6.2) is equivalent to

$$V\Delta_{\varphi}^{\alpha} = \Delta_{\psi}^{\alpha} V, \qquad \alpha \in \mathbb{C} \tag{6.3}$$

in the sense that $V\mathscr{D}(\Delta_{\varphi}^{\alpha}) \subseteq \mathscr{D}(\Delta_{\psi}^{\alpha})$ and (6.3) holds on $\mathscr{D}(\Delta_{\varphi}^{\alpha})$. Moreover, since $F(m^*) = F(m)^*$ $(m \in \mathscr{M})$ one has also $VS_{\varphi} = S_{\psi} V$ in the same sense. Thus (i) is equivalent to

$$VJ_{\varphi} = J_{\psi} V \Leftrightarrow V^{+}J_{\psi} = J_{\varphi} V^{+}. \tag{6.4}$$

Let now $v\colon \mathscr{N} \to \mathscr{M}$ be the bidual map of F, i.e. $v = j_{\varphi}^{-1} v' j_{\psi}$ and v' is the dual map of F in the sense of Proposition 3.1. Then one has, for each $n \in \mathscr{N}$

$$v(n)\Phi = J_{\varphi} V^{+} J_{\psi} n\Psi = V^{+} J_{\psi}^{2} n\Psi = V^{+} n\Psi,$$

which is equivalent to (6.2). That v is unique, normal, completely positive, identity preserving follows from the corresponding properties of dual maps (cf. Proposition 3.1). Thus (i) $\Rightarrow$ (ii). To prove the converse implication remark that (6.2) is equivalent to

$$v(n)\Phi = V^{+} n\Psi; \qquad n \in \mathscr{N}$$

which implies $V^+ S_\psi = S_\varphi V^+$ and, since S_φ, S_ψ are closed $S_\psi^+ V = VS_\varphi^+$. Thus $V\Delta_\varphi = VS_\varphi^+ S_\varphi = S_\psi^+ VS_\varphi = S_\psi^+ S_\psi V = \Delta_\psi V$ and, since $V\mathscr{D}(\Delta_\varphi) \subseteq \mathscr{D}(\Delta_\psi)$, relation $V\Delta_\varphi^{it} = \Delta_\psi^{it} V$ follows from a variant of a result of Bratteli and Robinson [7].

PROPOSITION 6.2. *In the notations of Sections* 2 *and* 3, *let* $E\colon \mathcal{A} \to \mathscr{B}$ *be the* φ*-conditional expectation. Then the following are equivalent*:

(i) $\sigma^t_{\mathscr{B}} E = E\sigma^t_{\mathcal{A}}$; $t \in \mathbb{R}$.

(ii) *There exists a* φ*-compatible embedding* $v'\colon \mathscr{B}' \hookrightarrow \mathcal{A}'$ *such that*

$$v'(b')\Phi = b'\Phi; \qquad b' \in \mathscr{B}'.$$

(iii) $\mathscr{B}'_+ \Phi \subseteq \mathcal{A}'_+ \Phi$.

(iv) E *is a norm one projection onto* $\mathscr{B}$.

Proof. Clearly (ii) $\Rightarrow$ (iii). If (iii) holds, then $\mathscr{B}'\Phi \subseteq \mathcal{A}'\Phi$ and, since Φ is separating for $\mathcal{A}'$, there exists a map $v'\colon \mathscr{B}' \hookrightarrow \mathcal{A}'$ uniquely defined by

$$v'(b')\Phi = b'\Phi; \qquad b' \in \mathscr{B}'.$$

Because of (iii), v' is positive. Let $F\colon \mathcal{A} \to \mathscr{B}$ be the dual map of v', uniquely defined, according to Proposition 3.1, by

$$\langle v'(b')\Phi, a\Phi\rangle = \langle b'\Phi, F(a)\Phi\rangle; \qquad b' \in \mathscr{B}',\ a \in \mathcal{A}$$

then by definition of v'

$$\langle v'(b')\Phi, a\Phi\rangle = \langle b'\Phi, a\Phi\rangle = \langle b'\Phi, Pa\Phi\rangle$$

therefore, by the cyclicity of Φ

$$F(a)\Phi = Pa\Phi; \qquad a \in \mathcal{A}$$

and therefore (iv) follows from Takesaki's theorem. The implication (iv) $\Rightarrow$ (i) is a simple consequence of Takesaki's Theorem. Finally if (i) holds, then by Proposition 6.1 there is a unique completely positive map $v\colon \mathscr{B} \to \mathcal{A}$ such that

$$v(b)\Phi = Ub\Phi; \qquad b \in \mathscr{B}$$

therefore, if $b \in \mathscr{B}_+$ one has $Ub\Phi = J_{\mathcal{A}} j_{\mathscr{B}}(b)\Phi = v(b)\Phi \subseteq \mathcal{A}_+ \Phi$ and this implies (iii) hence (iv). Therefore $U = P$ and, from this (ii) easily follows.

Remark 1. Condition (ii) in Proposition (6.2) is strictly stronger than $\mathscr{B}'\Phi \subseteq \mathcal{A}'\Phi$. In fact in the case of matrix algebras the latter condition is always fulfilled for a faithful φ, while the former is not, since in general the canonical φ-expectation is not a normal one projection.

Remark 2. Proposition 6.2 shows that in general $\sigma^t_{\mathscr{B}} E \neq E\sigma^t_{\mathcal{A}}$. A measure of how much this equality fails is provided by the following 1-cocycle (for the group $\Delta_{\mathcal{A}}^{-it}(\cdot)\Delta_{\mathcal{A}}^{+it}$)

$$U_t = U\Delta_{\mathscr{B}}^{-it}U^+ \Delta_{\mathcal{A}}^{it}. \tag{6.5}$$

One easily verifies that U_t is a 1-cocycle for $\Delta_{\mathcal{A}}^{-it}(\cdot)\,\Delta_{\mathcal{A}}^{it}$ and, denoting

$$\kappa_t^{-1}(a) = U_t a U_t^+ ; \qquad a \in \mathcal{A} \tag{6.6}$$

one has the formal identity

$$\sigma^t_{\mathscr{B}} E \kappa_t^{-1} = E\sigma^t_{\mathcal{A}}. \tag{6.7}$$

Identity (6.7) is only formal, since in general $\kappa_t^{-1}(\mathcal{A})$ is not contained in $\mathcal{A}$. However, in some cases it is possible to give a rigorous meaning to the identity:

$$\sigma^t_{\mathscr{B}} E = E\sigma^t_{\mathcal{A}} \kappa_t, \tag{6.8}$$

where $\kappa_t = U_t^+(\;)\,U_t$. More precisely:

PROPOSITION 6.3. *Assume that*

$$\mathscr{B}'\Phi \subseteq \mathcal{A}'\Phi; \tag{6.9}$$

then for each $t \in \mathbb{R}$ *there exists a map* $\kappa_t : \mathcal{A} \to \mathcal{A}$ *characterized by the property*

$$\kappa_t(a)\Phi = U_t^+ a\Phi; \qquad a \in \mathcal{A}. \tag{6.10}$$

Moreover, for each $t \in \mathbb{R}$ *and* $a, a_1, a_2 \in \mathcal{A}$

$$\sigma^t_{\mathscr{B}}(E(a)) = E(\sigma^t_{\mathcal{A}}(\kappa_t(a))), \tag{6.11}$$

$$\kappa_t(a_1) = \kappa_t(a_2) \Leftrightarrow E(a_1) = E(a_2). \tag{6.12}$$

Proof. Let $a \in A$, then:

$$U_t^+ a\Phi = \Delta_{\mathcal{A}}^{-it} U \Delta_{\mathscr{B}}^{it} U^+ a\Phi = \Delta_{\mathcal{A}}^{-it} J_{\mathcal{A}} j_{\mathscr{B}}(\sigma^t_{\mathscr{B}}(E(a)))\Phi.$$

Thus because of assumption (6.9),

$$U_t^+ a\Phi \in \Delta_{\mathcal{A}}^{-it} J_{\mathcal{A}} \mathscr{B}'\Phi \subseteq \Delta_{\mathcal{A}}^{-it} J_{\mathcal{A}}(\mathcal{A}'\Phi) = \Delta_{\mathcal{A}}^{-it}\mathcal{A}\Phi = \mathcal{A}\Phi.$$

thus for each $a \in \mathfrak{A}$ there exists an (unique since Φ is separating) element $\kappa_t(a) \in \mathfrak{A}$ satisfying (6.10). Moreover

$$E(\sigma^t_{\mathfrak{A}}(\kappa_t(a)))\Phi = U^+ \Delta^{it}_{\mathfrak{A}} \kappa_t(a)\Phi = U^+ \Delta^{it}_{\mathfrak{A}} \Delta^{-it}_{\mathfrak{A}} U \Delta^{it}_{\mathscr{B}} U^+ a\Phi$$
$$= \Delta^{it}_{\mathscr{B}} U^+ a\Phi$$

which is equivalent to (6.11). In a similar way one verifies that (6.12) is satisfied.

Remark 1. The cocycle U_t gives information on the relationship between the modular structures of a W^*-algebra $\mathfrak{A}$ and of a subalgebra $\mathscr{B}$. In some cases (especially when the fixed point algebra of E is rather large) decomposition (6.7) allows a simplified description of $\sigma^t_{\mathfrak{A}}$. For example, this happens for quantum Markov states on infinite tensor products of matrix algebras (cf. [3]).

Remark 2. In the assumptions of Proposition 6.3 one can prove that, for $a \in \mathfrak{A}$, the map $z \mapsto \kappa_z(a)$ is holomorphic for $0 \leqslant \mathrm{Re}(z) \leqslant \frac{1}{2}$ and that:

$$\kappa_{-i/2}(A) = E(a).$$

7. The φ-Expectation for Weights

In this section $\mathscr{M}$, $\mathscr{N}$ will be von Neumann algebras and $\mathscr{M} \supseteq \mathscr{N}$; φ denotes a faithful normal semi-finite weight on $\mathscr{M}_+$ whose restriction on $\mathscr{N}_+$, denoted φ_0, is also semi-finite. First of all let us recall some basic facts about weights and establish some notations [8, 11, 31]. Denote:

$$\mathfrak{m}_\varphi = \{x \in : \varphi(x^*x) < +\infty\} \tag{7.1}$$

$$\mathfrak{m}^*_\varphi = \{x \in \mathscr{M} : \varphi(xx^*) < +\infty\} \tag{7.2}$$

and let $\mathscr{M}(\varphi)$ be the set of all linear combinations of elements of the form $x^* \cdot y$ with $x, y \in \mathfrak{m}_\varphi$. $\mathscr{M}(\varphi)$ is an hereditary self-adjoint sub-algebra of $\mathscr{M}$ whose positive part $\mathscr{M}_+(\varphi)$ coincides with the set $\{x \in \mathscr{M}_+ : \varphi(x) < +\infty\}$ and the restriction of φ on $\mathscr{M}_+(\varphi)$ is extended uniquely to a linear functional on $\mathscr{M}(\varphi)$ which we, following the notations of [31], denote $\dot{\varphi}$. The set $\mathfrak{A} = \mathfrak{m}_\varphi \cap \mathfrak{m}^*_\varphi$ is an achieved left Hilbert algebra with left von Neumann algebra $\mathscr{L}(\mathfrak{A})$ isomorphic to $\mathscr{M}$ and scalar product

$$\langle a_1, a_2 \rangle = \dot{\varphi}(a_1^* \cdot a_2): \cdot a_1, a_2 \in \mathfrak{A}. \tag{7.3}$$

More precisely, denoting $\mathscr{H}$ the Hilbert space obtained as closure of $\mathfrak{A}$ with respect to the scalar product (7.3), $\pi : \mathfrak{A} \subseteq \mathscr{M} \to \mathscr{B}(\mathscr{H})$—the left multiplication ($\pi(a)\,a_1 = a \cdot a_1$; $a, a_1 \in \mathfrak{A}$), $\mathscr{L}(\mathfrak{A})$ the von Neumann algebra generated by $\pi(\mathfrak{A})$, one has that the map π can be extended to a W^*-isomorphism $\bar{\pi}: \mathscr{M} \to \mathscr{L}(\mathfrak{A})$. The involution in $\mathfrak{A}$ is the restriction of the involution $m \mapsto m^*$ ($m \in \mathscr{M}$) in $\mathscr{M}$; its closure will be $S_{\mathfrak{A}} = J_{\mathfrak{A}} \Delta_{\mathfrak{A}}^{1/2}$ and its adjoint $F_{\mathfrak{A}} = J_{\mathfrak{A}} \Delta_{\mathfrak{A}}^{-1/2}$.

By definition

$$\mathfrak{A}' = \{\xi \in \mathscr{D}(F_{\mathfrak{A}}): a \in \mathfrak{A} \mapsto \pi(a)\xi \text{ is continuous}\} \tag{7.4}$$

$$\pi'(\xi)a = \pi(a)\xi; \qquad a \in \mathfrak{A}; \quad \xi \in \mathfrak{A}'. \tag{7.5}$$

That $\mathfrak{A}$ is achieved means $(\mathfrak{A}')' = \mathfrak{A}'' = \mathfrak{A}$. In a similar way we introduce the corresponding structures associated to $\mathscr{N}$ and φ: $\mathfrak{n}_\varphi$, $\mathscr{N}(\varphi)$, $\mathscr{B} = \mathfrak{n}_\varphi \cap \mathfrak{n}_\varphi^*$, $\mathscr{B}'$, $\mathscr{L}(\mathscr{B})$, $\dot{\varphi}_0$, $\mathscr{K}$, $\pi_0: \mathscr{B} \to \mathscr{B}(\mathscr{K})$, π_0', $\bar{\pi}_0: \mathscr{N} \to \mathscr{L}(\mathscr{B})$, $S_{\mathscr{B}} = J_{\mathfrak{A}} \Delta_{\mathscr{B}}^{1/2}$, $F_{\mathscr{B}}$. Since $\mathscr{B} \subseteq \mathfrak{A}$ we can suppose that $\mathscr{K} \subseteq \mathscr{H}$ and the scalar product on $\mathscr{K}$ is the restriction of the one defined in $\mathscr{H}$ by (7.3).

With these notations let us define the partial isometry $U: \mathscr{H} \to \mathscr{H}$ with initial projection $\mathscr{K}$ such that

$$U\xi_0 = J_{\mathfrak{A}} J_{\mathscr{B}} \xi_0, \qquad \xi_0 \in \mathscr{K}. \tag{7.6}$$

Lemma 7.1. *In the above notations one has*:

$$U\mathscr{B}' \subset \mathfrak{A}'; \tag{7.7}$$

$$U\pi_0'(b') = \pi'(Ub')U, \qquad b' \in \mathscr{B}'; \tag{7.8}$$

$$Q\pi'(Ub') = \pi'(Ub')Q, \qquad b' \in \mathscr{B}' \tag{7.9}$$

(*where* $Q = UU^+$ *is the final projection of* U,

$$\|\pi'(Ub')\| = \|\pi_0'(b')\|. \tag{7.10}$$

Proof. We know [30, p. 23] that $J_{\mathscr{B}}\mathscr{B}' = \mathscr{B}'' = \mathscr{B}$. Therefore $J_{\mathfrak{A}} J_{\mathscr{B}} \mathscr{B} \subseteq J_{\mathfrak{A}} \mathfrak{A} = \mathfrak{A}'$; and this proves (7.7). Now let $b' \in \mathscr{B}'$. Then $\pi'(Ub')$ is well defined because of (7.7) and, if $b_1' \in \mathscr{B}'$ then:

$$\begin{aligned} U\pi_0'(b')\,b_1' &= Ub_1'b' = J_{\mathfrak{A}} \cdot J_{\mathscr{B}}(b_1'b) = J_{\mathfrak{A}}(J_{\mathscr{B}}b')(J_{\mathscr{B}}b_1') \\ &= (J_{\mathfrak{A}}J_{\mathscr{B}}b_1')(J_{\mathfrak{A}}J_{\mathscr{B}}b') = \pi'(Ub')\,Ub_1' \end{aligned}$$

and this proves (7.8) since $\mathscr{B}'$ is dense in $\mathscr{K}$, U has initial projection $\mathscr{K}$, and $\pi_0'(b')$ maps $\mathscr{K}$ into itself. From (7.8) we deduce that

$$U\pi_0'(b')\,U^+ = \pi'(Ub')Q; \qquad b' \in \mathscr{B}'. \tag{7.11}$$

In particular $\pi'(Ub')$ maps the final space of U into itself. To prove (7.9) we show that also $\pi'(Ub')^+$ has this property. Since $\pi'(Ub')^+ = \pi'(F_{\mathcal{A}}Ub')$ [30, p. 12] one has:

$$\begin{aligned}\pi'(Ub')^+ &= \pi'(F_{\mathcal{A}}Ub') = \pi'(F_{\mathcal{A}}J_{\mathcal{A}}J_{\mathcal{B}}b') \\ &= \pi'(J_{\mathcal{A}}S_{\mathcal{A}}J_{\mathcal{B}}b') = \pi'(J_{\mathcal{A}}S_{\mathcal{B}}J_{\mathcal{B}}b') \\ &= \pi'(J_{\mathcal{A}}J_{\mathcal{B}}F_{\mathcal{B}}b') = \pi'(Ub'^*)\end{aligned} \tag{7.12}$$

but $b'^* \in \mathcal{B}'$ and therefore in (7.11) we can substitute b'^* for b' and this, due to (7.12), implies our thesis. The equality (7.10) is an immediate consequence of the fact that $\mathcal{B} \subseteq \mathcal{A}$ the scalar product on $\mathcal{K}$ is the restriction on $\mathcal{K}$ of the scalar product on $\mathcal{H}$, and that, for $a \in \mathcal{A}$ and $b \in \mathcal{B}$, one has $\pi'(J_{\mathcal{A}}a) = J_{\mathcal{A}}\pi(a)J_{\mathcal{A}}$; $\pi'(J_{\mathcal{B}}b) = J_{\mathcal{B}}\pi_0(b)J_{\mathcal{B}}$.

Proposition 7.2. $U^+\mathcal{A} \subseteq \mathcal{B}$.

Proof. Clearly $U^+\mathcal{A} \subseteq \mathcal{K}$ = final space of U^+. We will show that for each $a \in \mathcal{A}$:

$$U^+a \in \mathcal{D}(S_{\mathcal{B}}), \tag{7.13}$$

$$b' \in \mathcal{B}' \mapsto \pi_0'(b')\, U^+a \text{ is continuous,} \tag{7.14}$$

i.e., that $U^+a \in (\mathcal{B}')' = \mathcal{B}$. Let $b' \in \mathcal{B}'$ then:

$$\begin{aligned}|\langle F_{\mathcal{B}}b', U^+a\rangle| &= |\langle J_{\mathcal{A}}J_{\mathcal{B}}F_{\mathcal{B}}b', a\rangle| \\ &= |\langle J_{\mathcal{A}}S_{\mathcal{B}}J_{\mathcal{B}}b', a\rangle| = |\langle J_{\mathcal{A}}S_{\mathcal{A}}J_{\mathcal{B}}b', a\rangle| \\ &= |\langle F_{\mathcal{A}}J_{\mathcal{A}}J_{\mathcal{B}}b', a\rangle| = |\langle Ub', S_{\mathcal{A}}a\rangle| \\ &\leqslant \|b'\| \cdot \|a^*\|\end{aligned}$$

hence $b' \in \mathcal{B}' \mapsto \langle F_{\mathcal{B}}b', U^+a\rangle$ extends to a bounded linear functional on $\mathcal{K}$, and this proves (7.13). Now, denoting P the orthogonal projection onto $\mathcal{K}$ one has that $P = U^+U$ and, using Lemma (7.1):

$$\begin{aligned}\|\pi_0'(b')\, U^+a\| &= \|U^+U\pi_0'(b')\, U^+a\| \\ &= \|U^+\pi'(Ub')a\| \leqslant \|\pi'(Ub')a\| \\ &= \|\pi(a)\,\mathcal{U}b'\| \leqslant \|\pi(a)\| \cdot \|b'\|\end{aligned}$$

and this proves (7.14)

Proposition 7.3. *For each* $a \in \mathcal{A}$ and $b' \in \mathcal{B}$:

$$U^+\pi(a)U \restriction \mathcal{K} = \pi_0(U^+a). \tag{7.15}$$

Proof. Using Lemma 7.1 we have that for each $b' \in \mathscr{B}'$

$$\begin{aligned} U^+\pi(a)\,Ub' &= U^+\pi'(Ub')a = U^+Q\pi'(Ub')a \\ &= U^+\pi'(Ub')\,UU^+a = U^+U\pi_0'(b')\,U^+a \\ &= P\pi_0'(b')\,U^+a = \pi_0'(b')\,U^+a \\ &= \pi_0(U^+a)\,b' \end{aligned}$$

and this proves (7.15) since $\mathscr{B}'$ is dense in $\mathscr{K}$.

From Proposition 7.3 we deduce that the map

$$a \in \mathfrak{A} \subseteq \mathscr{M} \mapsto \bar{\pi}_0^{-1}(U^+\pi(a)U) \in \mathscr{B} \subseteq \mathscr{N} \tag{7.16}$$

is well defined. It is completely positive and weakly continuous, being composed of maps which enjoy these properties. Therefore it can be extended to a completely positive weakly continuous map $E: \mathscr{M} \to \mathscr{N}$. Clearly $E(1) = 1$.

LEMMA 7.4. *For each $m \in \mathscr{M}_+$ such that $\varphi(m) < +\infty$ one has:*

$$\varphi(m) = \varphi_0(E(m)). \tag{7.17}$$

Proof. Let m be as above and denote $a = m^{1/2}$; then $a \in \mathfrak{A}$ and $\varphi(m) = \|a\|^2$. For $b' \in \mathscr{B}'$, using (7.10), we find

$$\begin{aligned} \langle \bar{\pi}(m)\,Ub', Ub' \rangle &= \|\pi(a)\,Ub'\|^2 \\ &= \|\pi'(Ub')a\|^2 \leqslant \|\pi'(Ub')\|^2 \cdot \|a\|^2 \\ &= \|\pi_0'(b')\|^2 \cdot \varphi(m). \end{aligned}$$

Since, by definition of E and (7.16), $\bar{\pi}_0(E(m)) = U^+\bar{\pi}(m)U$, (7.18) implies

$$\begin{aligned} \varphi(m) &\geqslant \sup\{\langle \bar{\pi}_0(E(m))\,b', b' \rangle : \|\pi_0'(b')\| \leqslant 1\} \\ &= \varphi_0(E(m)). \end{aligned}$$

Thus $E(m)^{1/2} \in \mathscr{B}$. Now, let (b'_α) be a net in $\mathscr{B}'$ which converges strongly to the identity and such that $\|\pi_0'(b'_\alpha)\| \leqslant 1$. Then, since $\pi_0(J_{\mathscr{B}}b'_\alpha) = J_{\mathscr{B}}\pi_0'(b'_\alpha)\,J_{\mathscr{B}}$, also the net $(\pi_0(J_{\mathscr{B}}b'_\alpha))$ has the same property and a similar argument plus the strong continuity of $\bar{\pi} \cdot \bar{\pi}_0^{-1}$ show that also the net $(\pi'(Ub'_\alpha))$ has this property. In particular $\pi'(Ub'_\alpha)a \in \mathscr{H}$ converges to a in norm, therefore:

$$\begin{aligned}\varphi(m) &= \|(a)\|^2 = \lim_\alpha \|\pi'(Ub'_\alpha)a\|^2 \\ &= \lim_\alpha \|\pi(a)\mathscr{U}b'_\alpha\| = \lim_\alpha \langle \bar{\pi}_0(E(m))\, b'_\alpha, b'_\alpha\rangle \\ &= \lim_\alpha \|\pi'_0(b'_\alpha)\, E(m)^{1/2}\|^2 \\ &= \|E(m)^{1/2}\|^2 = \varphi_0(E(m))\end{aligned}$$

and this proves (7.17).

Summing up our discussion we have:

THEOREM 7.5. *Let $\mathscr{M}$, $\mathscr{N}$ be von Neumann algebras with $\mathscr{N} \subseteq \mathscr{M}$, and let φ be a faithful normal semi-finite weight on $\mathscr{M}_+$ whose restriction φ_0 on $\mathscr{N}_+$ is semi-finite. Let $\mathfrak{A} \supseteq \mathscr{B}$ the achieved left Hilbert algebras associated to $\mathscr{M}$, φ and $\mathscr{N}$, φ_0 respectively; $J_{\mathfrak{A}}$, $J_{\mathscr{B}}$ the corresponding Tomita involutions and P the orthogonal projection from $\mathscr{H}$, i.e., the closure of $\mathfrak{A}$ with respect to the scalar product induced by φ—onto $\mathscr{K}$—i.e., the closure of $\mathscr{B}$ with respect to the same scalar product. Then the map*

$$a \in \mathfrak{A} \mapsto \pi_0^{-1}(J_{\mathscr{B}} P J_{\mathfrak{A}} \pi(a)\, J_{\mathfrak{A}} J_{\mathscr{B}} P) \in \mathscr{B}$$

is well defined and extends to a faithful normal completely positive identity preserving map satisfying

$$\varphi(m) = \varphi_0(E(m)); \qquad \varphi(m) < \infty; \qquad m \in \mathscr{M}_+. \tag{7.19}$$

Moreover the closure of $E(\mathfrak{A})$ is $\mathscr{K}$.

Proof. The existence of E follows from Proposition 7.3. Normality, complete positivity and $E(1) = 1$ follow from the explicit form (7.16) of E. That E is φ-compatible, i.e., (7.19), has been shown in Lemma (7.14) and the faithfulness of E is a consequence of this. Finally from (7.15) we see that $E(\mathfrak{A})$ is the image by the partial isometry U^+ of a dense set in $\mathscr{H}$ hence it is dense in $\mathscr{K}$ = closure of $\mathscr{B}$ = final space of U^+.

For a more detailed discussion of the properties of the φ-expectation associated to a weight see [8].

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